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# Diffeomorphism Symmetry in the Lagrangian Formulation of Gravity

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**Abstract** Starting from a knowledge of certain identities in the Lagrangian description, the diffeomorphism transformations of metric and connection are obtained for both the second order (metric) and the first order (Palatini) formulations of gravity. These transformations are found to be identical to the diffeomorphism transformations of the fields which establish a one-to-one mapping between the gauge and diffeomorphism symmetry.

**Keywords** Gravity · Diffeomorphism symmetry · Lagrangian analysis

## 1 Introduction

Classical description of gravity is best formulated by general relativity theory where gravity is treated as curvature of space time instead of an external force. In order to build up the whole formulation one needs to introduce two logically independent concepts, one is metric and the other is connection. Metric is a symmetric second rank covariant tensor which is used to define an invariant length on the manifold and the connection is introduced to map the vectors of different tangent spaces, which in turn is used to define the covariant derivative. Though these two concepts are entirely independent, in the standard version of general relativity one demands two conditions i.e. torsion free nature of connection and the vanishing covariant derivative of metric (metric compatibility condition) [1]. The violation of the first condition together with a general asymmetric metric leads to many theoretical results in which Einstein himself was very much interested [2, 3]. The second condition, in a special formulation of general relativity known as Palatini formulation, is not assumed a priori but derived as an equation of motion [4]. Though this formulation gives the same result as the standard general relativity in vacuum, it leads to distinct results when spinors are coupled to gravity [5].

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A relativistic description of classical gravity whether it is standard (metric) formulation or Palatini formulation is essentially based on the physical concept of general covariance principle. It is also known that the general theory of relativity is invariant under local Poincaré transformation. Thus, instead of taking the parameters of Poincaré transformations as some constants, if one considers them to be functions of space time, principle of equivalence emerges automatically.

But even without referring the Poincaré symmetry, the gauge transformations of the fields can be obtained from the Einstein Hilbert action where the presence of gauge symmetry is indicated by the first class constraints of the theory. Following Dirac's algorithm [6], generator of the gauge or diffeomorphism transformations can be constructed from a linear combination of the constraints which finally gives the diffeomorphism transformation of the metric field [7, 8]. An important tool in this formulation is the A-D-M (Arnowitt-Deser-Misner) decomposition [9] of space time where three spacelike surfaces evolve along a time like direction. In fact it is well known that this decomposition plays a central role in all Hamiltonian formulations of general relativity.

In the present paper we follow the Lagrangian description to obtain the gauge transformations of general relativity theory. By explicit computation we demonstrate that the Lagrangian gauge identities lead to the right diffeomorphism transformations of the fields which indicates a one-to-one correspondence between the gauge and the diffeomorphism invariance of the theory. This observation is made for both the second order metric gravity and the first order Palatini formalism of gravity.

Obtaining of gauge transformations systematically from the action of field variables is very important. As is well known, any symmetry transformation of the fields which keeps the action invariant is not necessarily a gauge transformation. Gauge symmetry means presence of extra degrees of freedom in the theory. The standard method to check a symmetry transformation is a gauge transformation or not is to find the corresponding Hamiltonian gauge generator. There is also an analogous approach in the Lagrangian formulation. In the present paper we follow the later approach to show the existence of certain identities which involve the Euler derivatives and the generators of gauge transformations. From these identities the diffeomorphism transformations of the basic fields are calculated in a systematic manner. As mentioned already, the Hamiltonian gauge theory when applied to general relativity needs one additional tool namely the A-D-M decomposition of space time. This decomposition is not required to study the Hamiltonian description of other theory like non-Abelian gauge theory. In this paper we follow the Lagrangian formulation and observe that the method which is used to analyze the non-Abelian gauge theory can be applied directly to the general relativity theory without adopting any extra tool. In this sense, the method discussed here is conceptually simpler than the Hamiltonian A-D-M decomposition.

This paper is organized as follows. In Sect. 2 we give a short description on the general method of analyzing the gauge symmetry in the Lagrangian frame work. Non-Abelian gauge theory, both in its second and first order versions are taken as examples in Sect. 3 to illustrate the method. The second order formulation of metric gravity is discussed in Sect. 4, whereas the first order Palatini formulation is analyzed in Sect. 5. Section 6 is left for discussions on the implications of present analysis and after that a short appendix is added.

## 2 General Formulation

It is a well known [10–14] fact that a theory containing gauge symmetry possesses an identity involving the various Euler derivatives of the theory. This identity is called the gauge identity from which the gauge transformations of the fields can be calculated. We briefly discuss this method in this section for a general gauge theory.

To study the dynamics of a field from an action principle we consider a general Lagrangian,

$$S = \int \mathrm{d}t\ \mathfrak{L} = \int \mathrm{d}^4 x\ \mathcal{L} \tag{1}$$

where the Lagrangian density, $\mathcal{L}$ is a function of the field variable $q(\mathbf{x}, t)$[1] and its space time derivatives of different orders. An arbitrary variation of this action is written as

$$\delta S = -\int \mathrm{d}^4 x\ \delta q(\mathbf{x}, t) L(\mathbf{x}, t). \tag{2}$$

For example, if Lagrangian density involves only up to first order derivative of the field

$$\mathfrak{L} = \int \mathrm{d}^3 x\ \ \mathcal{L}\left(q(\mathbf{x}, t),\ \partial_i q(\mathbf{x}, t),\ \partial_t q(\mathbf{x}, t)\right). \tag{3}$$

Euler derivative $L$ which appears in the right hand side of (2) is of the form [11]

$$L(\mathbf{x}, t) = \int \mathrm{d}^3 y \frac{\delta^2 \mathfrak{L}}{\delta \dot{q}(\mathbf{x}, t) \delta \dot{q}(\mathbf{y}, t)} \ddot{q}(\mathbf{y}, t) + \int \mathrm{d}^3 y \frac{\delta^2 \mathfrak{L}}{\delta q(\mathbf{x}, t) \delta \dot{q}(\mathbf{y}, t)} \dot{q}(\mathbf{y}, t) - \frac{\delta \mathfrak{L}}{\delta q(\mathbf{x}, t)}. \tag{4}$$

Applying the action principle in (2), the equations of motion are obtained by setting the Euler derivative to be zero i.e. $L = 0$. Now we vary the field $q$ in the following way

$$\delta q(\mathbf{x}, t) = \sum_{s=0}^{n} (-1)^s \int \mathrm{d}^3 \mathbf{z}\ \frac{\partial^s \eta(\mathbf{z}, t)}{\partial t^s} \rho_{(s)}(x, z) \tag{5}$$

with $\eta$ and $\rho$ being the parameter and generator, respectively, of the transformation. Under this variation of the field, the variation of the action is written from (2) as

$$\begin{aligned}
\delta S &= -\int \mathrm{d}^4 x \int \mathrm{d}^3 \mathbf{z}\ \eta(\mathbf{z}, t) \rho_{(0)}(x, z) L(\mathbf{x}, t) \\
&\quad - \int \mathrm{d}^4 x \sum_{s=1}^{n} (-1)^s \int \mathrm{d}^3 \mathbf{z}\ \frac{\partial}{\partial t} \left( \frac{\partial^{s-1} \eta(\mathbf{z}, t)}{\partial t^{s-1}} \right) \rho_{(s)}(x, z) L(\mathbf{x}, t) \\
&= -\int \mathrm{d}^4 z\ \eta(\mathbf{z}, t) \left( \int \mathrm{d}^3 \mathbf{x}\ \rho_{(0)}(x, z) L(\mathbf{x}, t) \right) \\
&\quad - \int \mathrm{d}^4 z\ \eta(\mathbf{z}, t) \left( \int \mathrm{d}^3 \mathbf{x}\ \frac{\partial}{\partial t} (\rho_{(1)}(x, z) L(\mathbf{x}, t)) \right) - \cdots.
\end{aligned} \tag{6}$$

We define a quantity [10, 11]

$$\Lambda(\mathbf{z}, t) = \left[ \sum_{s=0}^{n} \int \mathrm{d}^3 \mathbf{x}\ \frac{\partial^s}{\partial t^s} \left( \rho_{(s)}(x, z) L(\mathbf{x}, t) \right) \right] \tag{7}$$

[1] We use the notation $x$ for the four vector $x^\mu = (\mathbf{x}, t)$.

to write (6) in the form

$$\delta S = -\int \mathrm{d}^4 z\, \eta(\mathbf{z}, t)\Lambda(\mathbf{z}, t). \tag{8}$$

If the action does not change ($\delta S = 0$) under the field transformation (10) then it implies,

$$\Lambda(\mathbf{z}, t) = 0. \tag{9}$$

The last equality, which is called the gauge identity, must be true without use of any equation of motion. For more than one gauge invariance, (5) and (7) will be generalized as,

$$\delta q^\alpha(\mathbf{x}, t) = \sum_{s=0}^{n} (-1)^s \int \mathrm{d}^3\mathbf{z}\, \frac{\partial^s \eta^b(\mathbf{z}, t)}{\partial t^s} \rho^{\alpha b}_{(s)}(x, z) \tag{10}$$

and

$$\Lambda^a(\mathbf{z}, t) = \left[\sum_{s=0}^{n} \int \mathrm{d}^3\mathbf{x}\, \frac{\partial^s}{\partial t^s} \left(\rho^{\alpha a}_{(s)}(x, z) L_\alpha(\mathbf{x}, t)\right)\right]. \tag{11}$$

## 3 An Example: Non-Abelian Gauge Theory

Here we study a particular example to illustrate the general formalism discussed in the previous section. We take the standard second order action of the non-Abelian gauge theory,

$$S = \int \mathrm{d}^4 x \left[-\frac{1}{2}\mathrm{Tr} F_{\mu\nu}(x) F^{\mu\nu}(x)\right] \tag{12}$$

where the generators ($T^a$) of the gauge group satisfy[2]

$$[T^a, T^b] = i f^{abc} T^c. \tag{13}$$

The Euler derivatives $L^{\mu a}$ gives the equations of motion

$$L^{\mu a} = -(\mathcal{D}_\sigma F^{\sigma\mu})^a = 0. \tag{14}$$

Here the covariant derivative $\mathcal{D}$ is defined in the adjoint representation,

$$\mathcal{D}_\mu \xi = \partial_\mu \xi + ig[A_\mu, \xi]. \tag{15}$$

In order to obtain the gauge transformations we first find the gauge identity. It is given by,

$$\Lambda^a = -\left(\mathcal{D}^\mu L_\mu\right)^a = 0. \tag{16}$$

Comparing this identity with the general form (11) we get the following generators [12]

$$\rho^{b0a}_{(0)}(x, z) = -g f^{abc} \delta^3(\mathbf{x} - \mathbf{z}) A^c_0(x), \tag{17}$$

[2] We choose the representation where the structure constant $f^{abc}$ is antisymmetric in all indexes and $\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$.

$$\rho^{b0a}_{(1)}(x,z) = -\delta^{ab}\delta^3(\mathbf{x}-\mathbf{z}), \tag{18}$$

$$\begin{aligned}\rho^{bia}_{(0)}(x,z) &= -\delta^{ab}\partial^{i\mathbf{z}}\delta^3(\mathbf{x}-\mathbf{z}) \\ &\quad - gf^{abc}\delta^3(\mathbf{x}-\mathbf{z})A^{ic}(x).\end{aligned} \tag{19}$$

Using these expressions of the generators, we now obtain the gauge transformations of the $A^\mu$ field from (10)

$$\delta A^{\mu a} = \partial^\mu \eta^a - gf^{abc}A^{\mu b}\eta^c = (\mathcal{D}^\mu \eta)^a. \tag{20}$$

We now consider the first order action of the non-Abelian gauge theory

$$S = \int \mathrm{d}^4x \, \frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} - \mathrm{Tr}F_{\mu\nu}(\partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]) \tag{21}$$

where $A^\mu$ and $F^{\mu\nu}$ are treated as independent fields. Following the same method, the Euler derivatives are calculated which lead to the equations of motion

$$L^\mu \equiv \mathcal{D}_\nu F^{\nu\mu} = 0, \tag{22}$$

$$L^{\mu\nu} \equiv \frac{1}{2}[F^{\mu\nu} - (\partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu])] = 0. \tag{23}$$

A gauge identity is also found similar to the second order formulation

$$\mathcal{D}_\nu L^\nu + ig[F^{\mu\nu}, L_{\mu\nu}] = 0. \tag{24}$$

Expanding (11) to explicitly write the two distinct Euler derivatives, we obtain the gauge identity as,

$$\begin{aligned}\Lambda^a(\mathbf{z},t) &= \sum_s \int \mathrm{d}^3\mathbf{x}\, \frac{\partial^s}{\partial t^s}\left(\rho^{b\mu a}_{(s)}(x,z)L^b_\mu(\mathbf{x},t)\right) \\ &\quad + \sum_s \int \mathrm{d}^3\mathbf{x}\, \frac{\partial^s}{\partial t^s}\left(\rho^{b\mu\nu a}_{(s)}(x,z)L^b_{\mu\nu}(\mathbf{x},t)\right) = 0.\end{aligned} \tag{25}$$

Comparing this with (24) we obtain the generators (17, 18, 19) together with

$$\rho^{b\mu\nu a}_{(0)}(x,z) = -gf^{abc}\delta^3(\mathbf{x}-\mathbf{z})F^{\mu\nu c}(x). \tag{26}$$

The gauge transformations of the field $A_\mu$ and $F^{\mu\nu}$ are now obtained from the relation (10)

$$\delta A^\mu = (\mathcal{D}^\mu \eta), \tag{27}$$

$$\delta F^{\mu\nu a}(x) = gf^{abc}\eta^b F^{\mu\nu c}. \tag{28}$$

In this way we obtain the gauge variation of the gauge field ($A^\mu$) and the field strength ($F^{\mu\nu}$) in an independent manner for the first order formulation.

## 4 Metric Formulation

So far we were discussing the general method of analyzing the gauge symmetry in the Lagrangian framework and took non-Abelian gauge theory as our example to elaborate the whole procedure for both the second order and the first order descriptions. Now we are in a position to study the diffeomorphism symmetry of the general theory of relativity. In this section we study the second order formulation which is usually called the metric formulation. The less studied first order formulation i.e. the Palatini formulation, will be considered in the next section.

The Einstein-Hilbert action which describes the metric formulation of gravity is given by,

$$S = \int d^4x \mathcal{L}(g)$$
$$= \int d^4x \sqrt{-g} R = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(g) \tag{29}$$

where $R_{\mu\nu}$ is the Ricci tensor

$$R_{\mu\nu} = \Gamma^\lambda_{\nu\mu,\lambda} - \Gamma^\lambda_{\lambda\mu,\nu} + \Gamma^\lambda_{\nu\mu}\Gamma^\sigma_{\sigma\lambda} - \Gamma^\sigma_{\lambda\mu}\Gamma^\lambda_{\nu\sigma}. \tag{30}$$

The metric compatibility condition

$$\nabla_\rho g_{\mu\nu} = 0 \tag{31}$$

defines the Christoffel connection in terms of the metric components,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}). \tag{32}$$

Varying the action (29) with respect to the metric $g_{\mu\nu}$ we get the Euler derivative $L_{\mu\nu}$ i.e.

$$\delta S = \int L^{\mu\nu} \delta g_{\mu\nu} \tag{33}$$

where the explicit form of $L^{\mu\nu}$ is written as,

$$L^{\mu\nu} = \sqrt{-g} G^{\mu\nu} = \sqrt{-g}\left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \tag{34}$$

leading to the usual Einstein's equation, $L^{\mu\nu} = 0$. Now to find the gauge identity we recall the Bianchi identity [15]

$$\nabla_\eta R_{\lambda\mu\nu\kappa} + \nabla_\nu R_{\lambda\mu\kappa\eta} + \nabla_\kappa R_{\lambda\mu\eta\nu} = 0 \tag{35}$$

which follows from the definition of the Riemann tensor

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma} R^\sigma{}_{\mu\nu\kappa} = g_{\lambda\sigma}(\Gamma^\sigma_{\mu\kappa,\nu} - \Gamma^\sigma_{\mu\nu,\kappa} + \Gamma^\eta_{\mu\kappa}\Gamma^\sigma_{\nu\eta} - \Gamma^\eta_{\mu\nu}\Gamma^\sigma_{\kappa\eta}). \tag{36}$$

Contracting $\lambda$ with $\nu$ and $\mu$ with $\kappa$, in (35), using (31) we get

$$\nabla_\mu G^{\mu\nu} = 0. \tag{37}$$

This contracted Bianchi which means that Einstein tensor $G^{\mu\nu}$ is divergence free is referred as the gauge identity in [16]. But the Euler derivative we defined in (34) is not $G^{\mu\nu}$ but $\sqrt{-g} G^{\mu\nu}$. So we take our gauge identity as,

$$\Lambda_\alpha \equiv 2\nabla_\beta L^\beta_\alpha = 0. \tag{38}$$

The extra factor 2 is introduced for later convenience. In order to write the above (38) in a more convenient way we note that, the definition of $\Gamma$ (32) can be used to write the divergence of Einstein tensor

$$\nabla_\mu G^\mu_\nu = \partial_\mu G^\mu_\nu + \Gamma^\mu_{\mu\alpha} G^\alpha_\nu - \Gamma^\alpha_{\mu\nu} G^\mu_\alpha \tag{39}$$

in the following form

$$\nabla_\mu G^\mu_\nu = \left( \partial_\mu G^\mu_\nu + \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta} G^\alpha_\nu - \frac{1}{2} G^{\mu\beta} \partial_\nu g_{\beta\mu} \right). \tag{40}$$

Now using (31) and (40) we write the gauge identity (38) in the form

$$\begin{aligned}\Lambda_\nu &= 2\nabla_\mu L^\mu_\nu = 2\nabla_\mu \sqrt{-g} G^\mu_\nu \\ &= 2\sqrt{-g} \left( \partial_\mu G^\mu_\nu + \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta} G^\alpha_\nu - \frac{1}{2} G^{\mu\beta} \partial_\nu g_{\beta\mu} \right) \\ &= 2\partial_\mu \sqrt{-g} G^\mu_\nu - \partial_\nu g_{\alpha\beta} \sqrt{-g} G^{\alpha\beta} \\ &= 2\partial_\mu L^\mu_\nu - \partial_\nu g_{\alpha\beta} L^{\alpha\beta}\end{aligned} \tag{41}$$

where we have used the important relation

$$\partial_\mu g = g g^{\alpha\beta} \partial_\mu g_{\alpha\beta}. \tag{42}$$

In the metric formulation of gravity the analogy of (11) is expressed as,

$$\Lambda_\alpha(z) = \sum_{s=0}^{n} \int \mathrm{d}^3\mathbf{x} \frac{\partial^s}{\partial t^s} \left( \rho_{\mu\nu\alpha(s)}(x, z) L^{\mu\nu}(x) \right). \tag{43}$$

Comparing this equation with the identity (41), we get the following expressions for the non vanishing generators

$$\rho_{00\mu(0)} = -\partial_\mu g_{00} \delta(x - z), \tag{44}$$

$$\rho_{00\mu(1)} = 2 g_{0\mu} \delta(x - z), \tag{45}$$

$$\rho_{0i\mu(0)} = -\partial_\mu g_{0i} \delta(x - z) + \partial^z_i \left( g_{0\mu} \delta(x - z) \right), \tag{46}$$

$$\rho_{0i\mu(1)} = g_{\mu i} \delta(x - z), \tag{47}$$

$$\rho_{ij\mu(0)} = -\partial_\mu g_{ij} \delta(x - z) + \partial^z_j \left( g_{i\mu} \delta(x - z) \right) + \partial^z_i \left( g_{j\mu} \delta(x - z) \right). \tag{48}$$

After getting the expressions for the generators it is now straightforward to calculate the gauge transformations of the metric $g_{\mu\nu}$ under the Lagrangian gauge symmetry. Later these transformations will be identified as the diffeomorphism transformations of the metric. We write (10) for this formulation of gravity as

$$\delta g_{\mu\nu}(x) = \sum_{s=0}^{n} (-1)^s \int \mathrm{d}^3\mathbf{z} \, \frac{\partial^s \varepsilon^\alpha(z)}{\partial t^s} \rho_{\mu\nu\alpha(s)}(x, z). \tag{49}$$

From the above equation we write $\delta g_{00}$ as,

$$\delta g_{00}(x) = \int d^3\mathbf{z} \Big[ \varepsilon^0(z)\rho_{000(0)}(x,z) + \varepsilon^k(z)\rho_{00k(0)}(x,z) - \frac{\partial\varepsilon^0}{\partial t}(z)\rho_{000(1)}(x,z) - \frac{\partial\varepsilon^k}{\partial t}(z)\rho_{000(1)}(x,z)\Big]. \tag{50}$$

Using the generators (44, 45) in the above equation we get

$$\delta g_{00} = -\partial_0 g_{00}\varepsilon^0 - 2g_{00}\partial_0\varepsilon^0 - 2g_{0k}\partial_0\varepsilon^k - \partial_k g_{00}\varepsilon^k. \tag{51}$$

Similarly all other components of the metric $g_{\mu\nu}$ can be calculated. Combining everything we write the variation as,

$$\delta g_{\mu\nu} = -\partial_\alpha g_{\mu\nu}\varepsilon^\alpha - g_{\mu\alpha}\partial_\nu\varepsilon^\alpha - g_{\alpha\nu}\partial_\mu\varepsilon^\alpha. \tag{52}$$

Above result expresses the gauge transformation of the metric field $g_{\mu\nu}$. The variation of the inverse metric $g^{\mu\nu}$ is obtained easily from the above equation (52) by observing that

$$\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}. \tag{53}$$

The expression of $\delta g^{\mu\nu}$ we thus find, is written as,

$$\delta g^{\mu\nu} = -\partial_\alpha g^{\mu\nu}\varepsilon^\alpha + g^{\mu\alpha}\partial_\alpha\varepsilon^\nu + g^{\alpha\nu}\partial_\alpha\varepsilon^\mu. \tag{54}$$

We next calculate the gauge transformation of the connection from its definition (32). Making use of (52) and (54) we find it to be

$$\delta\Gamma^\rho_{\mu\nu} = -\varepsilon^\alpha\partial_\alpha\Gamma^\rho_{\mu\nu} + \Gamma^\alpha_{\mu\nu}\partial_\alpha\varepsilon^\rho - \Gamma^\rho_{\mu\alpha}\partial_\nu\varepsilon^\alpha - \Gamma^\rho_{\alpha\nu}\partial_\mu\varepsilon^\alpha - \partial_\mu\partial_\nu\varepsilon^\rho. \tag{55}$$

The diffeomorphism transformations of the metric field and the connection are derived in Appendix ((88) and (91)) from a point of view of infinitesimal general coordinate transformation. The results are identical with (52) and (55) which shows a one-to-one correspondence between the gauge and the diffeomorphism symmetry.

## 5 Palatini Formulation

Similar to the first order formulation of the non-Abelian gauge theory there is also a first order version of the Einstein-Hilbert action [17]

$$S = \int d^4x\,\mathcal{L}(g,\Gamma) = \int d^4x\sqrt{-g}\,g^{\mu\nu}R_{\mu\nu}(\Gamma) \tag{56}$$

where $g$, $\Gamma$ are now treated as independent field variables. The definition of $R_{\mu\nu}$ in terms of $\Gamma$ is same as (30). In our present analysis we take $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ as symmetric in $\mu$ and $\nu$. Variation of the above action with respect to $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ gives

$$\delta S = \int L^{\mu\nu}\delta g_{\mu\nu} + \int E^{\mu\nu}_\sigma\delta\Gamma^\sigma_{\mu\nu}. \tag{57}$$

The expressions of the Euler derivatives $L^{\mu\nu}$ and $E_\sigma^{\mu\nu}$ are given below

$$L^{\mu\nu} = \sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right), \tag{58}$$

$$\begin{aligned} E_\sigma^{\mu\nu} &= (\sqrt{-g}g^{\mu\nu})_{,\sigma} + \sqrt{-g}g^{\rho\nu}\Gamma^\mu_{\rho\sigma} + \sqrt{-g}g^{\mu\rho}\Gamma^\nu_{\rho\sigma} - \sqrt{-g}g^{\mu\nu}\Gamma^\rho_{\rho\sigma} \\ &\quad - \frac{1}{2}[(\sqrt{-g}g^{\mu\beta})_{,\beta} + \sqrt{-g}g^{\rho\beta}\Gamma^\mu_{\rho\beta}]\delta^\nu_\sigma \\ &\quad - \frac{1}{2}[(\sqrt{-g}g^{\nu\beta})_{,\beta} + \sqrt{-g}g^{\rho\beta}\Gamma^\nu_{\rho\beta}]\delta^\mu_\sigma \end{aligned} \tag{59}$$

leading to the equations of motion,

$$L^{\mu\nu} = 0, \tag{60}$$

$$E_\sigma^{\mu\nu} = 0. \tag{61}$$

The metric compatibility condition (31) can be derived from the second equation of motion [16] except for two dimension. The issues related to two dimension may be found in [17]. After obtaining the Euler derivatives we now give the gauge identity for the Palatini formulation

$$\begin{aligned} \Lambda_\alpha &\equiv -L^{\mu\nu}\partial_\alpha g_{\mu\nu} + 2\partial_\nu(g^{\nu\sigma}L_{\alpha\sigma}) \\ &\quad - E_\rho^{\mu\nu}\partial_a\Gamma^\rho_{\mu\nu} - \partial_\rho(E_\alpha^{\mu\nu}\Gamma^\rho_{\mu\nu}) + 2\partial_\mu(E_\rho^{\mu\nu}\Gamma^\rho_{\alpha\nu})\partial_\mu\partial_\nu E_\alpha^{\mu\nu} = 0. \end{aligned} \tag{62}$$

Note that in the metric formulation $E_\sigma^{\mu\nu}$ is identically zero and (62) reduces to the identity (41) which came from the double contraction of the Bianchi identity (35). It is worthwhile to mention that there is also a Bianchi identity [16] valid for Palatini formulation

$$\nabla_\eta R^\lambda{}_{\mu\nu\kappa} + \nabla_\nu R^\lambda{}_{\mu\kappa\eta} + \nabla_\kappa R^\lambda{}_{\mu\eta\nu} = 0. \tag{63}$$

The relation (43) is now rewritten to include the independent Euler derivatives as,

$$\Lambda_\alpha(z) = \sum_{s=0}^{n}\int d^3\mathbf{x}\frac{\partial^s}{\partial t^s}\left(\rho_{\mu\nu\alpha(s)}(x,z)L^{\mu\nu}(x) + \rho^\sigma_{\mu\nu\alpha(s)}(x,z)E_\sigma^{\mu\nu}(x)\right). \tag{64}$$

Using the explicit expressions for the Euler derivatives (58, 59), the generators are read off by comparing (62) with (64), naturally, the generators $\rho_{\mu\nu\alpha(s)}$ are identical to the generators of the metric formulation, exactly as happened for the gauge theory. The expressions of the other generators are given below

$$\rho^0_{00\mu(0)} = -\partial_\mu\Gamma^0_{00}\delta(x-z) - \delta^0_\mu\partial^z_m\left(\Gamma^m_{00}\delta(x-z)\right), \tag{65}$$

$$\rho^0_{000(1)} = \Gamma^0_{00}\delta(x-z), \tag{66}$$

$$\rho^0_{00k(1)} = 2\Gamma^0_{0k}\delta(x-z), \tag{67}$$

$$\rho^0_{000(2)} = \rho^k_{00m(2)} = -\delta(x-z), \tag{68}$$

$$\rho^0_{0i\mu(0)} = -\partial_\mu\Gamma^0_{0i}\delta(x-z) - \delta^0_\mu\partial^z_m\left(\Gamma^m_{0i}\delta(x-z)\right) + \partial^z_i\left(\Gamma^0_{0\mu}\delta(x-z)\right), \tag{69}$$

$$\rho^0_{0ik(1)} = \Gamma^0_{ki}\delta(x-z), \tag{70}$$

$$\rho^0_{0i0(1)} = -\partial^z_i \delta(x-z), \tag{71}$$

$$\rho^0_{ij\mu(0)} = -\partial_\mu \Gamma^0_{ij}\delta(x-z) + \partial^z_j\left(\Gamma^0_{i\mu}\delta(x-z)\right) + \partial^z_i\left(\Gamma^0_{\mu j}\delta(x-z)\right) - \delta^0_\mu \partial^z_m\left(\Gamma^m_{ij}\delta(x-z)\right) - \delta^0_\mu \partial^z_i\partial^z_j\delta(x-z), \tag{72}$$

$$\rho^k_{00\mu(0)} = -\partial_\mu \Gamma^k_{00}\delta(x-z) - \partial^z_p\left(\Gamma^p_{00}\delta(x-z)\right)\delta^k_\mu, \tag{73}$$

$$\rho^k_{00\mu(1)} = -\Gamma^0_{00}\delta^k_\mu\delta(x-z) + 2\Gamma^k_{0\mu}\delta(x-z), \tag{74}$$

$$\rho^k_{0i\mu(0)} = -\partial_\mu \Gamma^k_{0i}\delta(x-z) - \delta^k_\mu\partial^z_p\left(\Gamma^p_{0i}\delta(x-z)\right) + \partial^z_i\left(\Gamma^k_{0\mu}\delta(x-z)\right), \tag{75}$$

$$\rho^k_{0i\mu(1)} = -\delta^k_\mu\Gamma^0_{0i}\delta(x-z) + \Gamma^k_{\mu i}\delta(x-z) - \partial^z_i\delta(x-z)\delta^k_\mu, \tag{76}$$

$$\rho^k_{ij\mu(0)} = -\partial_\mu \Gamma^k_{ij}\delta(x-z) + \partial^z_i\left(\Gamma^k_{j\mu}\delta(x-z)\right) + \partial^z_j\left(\Gamma^k_{i\mu}\delta(x-z)\right) - \delta^k_\mu\partial^z_p\left(\Gamma^p_{ij}\delta(x-z)\right) - \delta^k_\mu\partial^z_i\partial^z_j\delta(x-z), \tag{77}$$

$$\rho^\nu_{ij\mu(1)} = -\delta^\nu_\mu\Gamma^0_{ij}\delta(x-z). \tag{78}$$

These generators are now used to calculate the diffeomorphism transformation of the fields $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ from the following equations

$$\delta g_{\mu\nu}(x) = \sum_{s=0}^{n}(-1)^s \int \mathrm{d}^3\mathbf{z}\frac{\partial^s\varepsilon^\alpha}{\partial t^s}(z)\rho_{\mu\nu\alpha(s)}(x,z), \tag{79}$$

$$\delta\Gamma^\rho_{\mu\nu}(x) = \sum_{s=0}^{n}(-1)^s \int \mathrm{d}^3\mathbf{z}\frac{\partial^s\varepsilon^\alpha}{\partial t^s}(z)\rho^\rho_{\mu\nu\alpha(s)}(x,z). \tag{80}$$

Since the generators $\rho_{\mu\nu\alpha(s)}$ are same as the generators of the metric formulation of gravity, the gauge variation of the metric is same as (52). From (80) we write $\delta\Gamma^0_{00}$ explicitly

$$\delta\Gamma^0_{00}(x) = \int \mathrm{d}^3\mathbf{z}\left[\varepsilon^0(z)\rho^0_{000(0)}(x,z) + \varepsilon^k(z)\rho^0_{00k(0)}(x,z) - \frac{\partial\varepsilon^0}{\partial t}(z)\rho^0_{000(1)}(x,z) - \frac{\partial\varepsilon^k}{\partial t}(z)\rho^0_{00k(1)}(x,z) + \frac{\partial^2\varepsilon^0}{\partial t^2}(z)\rho^0_{000(2)}(x,z)\right] \tag{81}$$

using (65, 66, 67, 68) in the above equation we get

$$\delta\Gamma^0_{00} = -\varepsilon^\alpha\partial_\alpha\Gamma^0_{00} + \Gamma^\alpha_{00}\partial_\alpha\varepsilon^0 - 2\Gamma^0_{0\alpha}\partial_0\varepsilon^\alpha - \partial^2_0\varepsilon^0. \tag{82}$$

All other components of the connection can be calculated in a similar manner. The results thus obtained are written in the following way

$$\delta\Gamma^\rho_{\mu\nu} = -\varepsilon^\alpha\partial_\alpha\Gamma^\rho_{\mu\nu} + \Gamma^\alpha_{\mu\nu}\partial_a\varepsilon^\rho - \Gamma^\rho_{\mu\alpha}\partial_\nu\varepsilon^\alpha - \Gamma^\rho_{\alpha\nu}\partial_\mu\varepsilon^\alpha - \partial_\mu\partial_\nu\varepsilon^\rho. \tag{83}$$

Thus in the Palatini formulation, the gauge transformation of the connection is derived independently from the gauge variation of the metric. The same result is derived in Appendix (91) using the transformation of the metric (52).

## 6 Discussions

We have studied the gauge symmetries of the general relativity theory for both the second order (metric) and the first order (Palatini) formulations of gravity. Some identities were obtained from which the generators of the transformations were found. The gauge transformations of the metric and the connection for both approaches were systematically derived. The results thus obtained were identical to the diffeomorphism transformations and compatible with each other. In this regard the present study of this paper can be considered as an alternative to Hamiltonian analysis.

A point worth mentioning is that, although the symmetry transformation rules for the metric and the Christoffel connection are well known, explicit computations of those rules within the Lagrangian framework, for the general relativity theory either in the metric or in the Palatini formulations are completely new. Since the method we follow here is very general one can apply this to other formulations of gravity where the action is not obtained directly from the general covariance principle but rather as a limiting case of some other theory. For example in string theory higher order corrections to the Einstein–Hilbert action is found in the semi classical regime. There are also classical theories like Lovelock gravity [18] where the Lagrangian density is a general function of the Riemann tensor not just simply the Ricci scalar. For such theories our formulation is still applicable to find the gauge transformations of the basic fields. Even in the context of classical general relativity theory, symmetry analysis has been done thoroughly within the Hamiltonian framework [7, 8]. But the Lagrangian formulation which is known to complement the Hamiltonian formulation lacked such symmetry analysis. In the present paper we have discussed the gauge symmetry of general relativity theory within the Lagrangian framework. The constraints of the theory which are important for the Hamiltonian gauge analysis can also be obtained here from the time component of the Euler derivatives. The gauge transformations of the metric obtained here, when written in terms of the lapse and shift variables match exactly with the results found from A-D-M decomposition technique [8]. Thus, in this paper we found consistent results which in our belief fill the existing gap between two approaches (Lagrangian and Hamiltonian) in the context of diffeomorphism symmetry.

A main result of our paper is that, the techniques of gauge theory which are used to study the symmetries of a gauge system are applied successfully to describe the symmetries of general relativity theory. The frequently used statement 'gravity is a gauge theory' is thus established here in the Lagrangian version of the theory. Regarding the equivalence of the first order and the second order formulations of gravity we observe that, though the diffeomorphism transformations for both the formulations are compatible with each other, there is one important difference hidden in their respective gauge identities. The gauge identity of second order metric formulation is shown to follow from a more general Bianchi identity as shown in Sect. 4 whereas the gauge identity of the other (first order) formulation, mentioned in Sect. 5 does not follow from the Bianchi identity of the first order Palatini formulation.

It is worthwhile to pursue the consequence of symmetry analysis for other approaches of gravity since it is a well known fact that Einstein-Hilbert action is not the unique action which is invariant under general coordinate transformation. Even for a more general Lovelock gravity formulation [18] it is shown that classical results obtained from metric and Palatini formulations are completely equivalent [19]. In presence of different other terms in an action (higher order gravity theory [20–22]) apart from the standard Einstein Hilbert term, metric and Palatini formulations are not equivalent in general [23, 24]. For such an action the present symmetry studies may give new insights about the problem.

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## Appendix

Here we give a brief derivation of the diffeomorphism transformation of the fields. Under a general coordinate transformation, $g_{\mu\nu}$ transforms as a covariant second rank tensor i.e.

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{84}$$

Now we consider an infinitesimal coordinate transformation

$$x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x) \tag{85}$$

under which, we write from (84)

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - g_{\mu\alpha}(x)\partial_\nu\varepsilon^\alpha - g_{\alpha\nu}(x)\partial_\mu\varepsilon^\alpha + \mathcal{O}(\varepsilon^2). \tag{86}$$

A Taylor expansion of the r.h.s. of the above equation, using (85), gives

$$g'_{\mu\nu}(x') = g'_{\mu\nu}(x+\varepsilon) = g'_{\mu\nu}(x) + \partial_\alpha g_{\mu\nu}\varepsilon^\alpha + \mathcal{O}(\varepsilon^2). \tag{87}$$

Combining (86) and (87) we get

$$\delta g_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -\partial_\alpha g_{\mu\nu}\varepsilon^\alpha - g_{\mu\alpha}\partial_\nu\varepsilon^\alpha - g_{\alpha\nu}\partial_\mu\varepsilon^\alpha. \tag{88}$$

This can be written in a covariant notation also [25]

$$\delta g_{\mu\nu} = -\nabla_\mu\varepsilon_\nu - \nabla_\nu\varepsilon_\mu \tag{89}$$

where the definition of connection (32) has been used. In a similar way the variation of the inverse metric $g^{\mu\nu}$ is also obtained

$$\delta g^{\mu\nu} = -\partial_\alpha g^{\mu\nu}\varepsilon^\alpha + g^{\mu\alpha}\partial_\alpha\varepsilon^\nu + g^{\alpha\nu}\partial_\alpha\varepsilon^\mu. \tag{90}$$

Using (88) and (90) it is now straightforward to calculate the diffeomorphism transformation of the connection from (32). We find it to be

$$\delta\Gamma^\rho_{\mu\nu} = -\varepsilon^\alpha\partial_\alpha\Gamma^\rho_{\mu\nu} + \Gamma^\alpha_{\mu\nu}\partial_\alpha\varepsilon^\rho - \Gamma^\rho_{\mu\alpha}\partial_\nu\varepsilon^\alpha - \Gamma^\rho_{\alpha\nu}\partial_\mu\varepsilon^\alpha - \partial_\mu\partial_\nu\varepsilon^\rho. \tag{91}$$

This equation together with (88) give the gauge variation of the two most important quantities of general theory of relativity.

## References


1. Carroll, S.M.: An Introduction to General Relativity: Space time and Geometry. Addison Wesley, Reading (2003)
2. Einstein, A., Straus, E.G.: A generalization of the relativistic theory of gravitation, 2. Ann. Math. **47**, 731 (1946)
3. Einstein, A., Kaufman, B.: A new form of the general relativistic field equations. Ann. Math. **62**, 128 (1955)
4. Palatini, A.: Deduzione invariantiva delle equazioni gravitazionali dal. principio di Hamilton. Rend. Circ. Mat. Palermo **43**, 203 (1919)
5. Deser, S., Zumino, B.: Consistent supergravity. Phys. Lett. **B 62**, 335 (1976)
6. M Dirac, P.A.: Lectures on Quantum Mechanics. Yeshiva University Press, New York (1964)

7. Castellani, L.: Symmetries in constrained Hamiltonian systems. Ann. Phys. **143**, 357–371 (1982)
8. Mukherjee, P., Saha, A.: Gauge invariances vis-a-vis diffeomorphisms in second order metric gravity. arXiv:0705.4358 [hep-th]
9. Arnowitt, R., Deser, S., Misner, C.W.: Gravitation: an Introduction to Current Research. Wiley, New York (1962). L. Witten (ed.)
10. Gitman, D.M., Tyutin, I.V.: Quantization of Fields with Constraints. Springer, Berlin (1990)
11. Shirzad, A.: Gauge symmetry in Lagrangian formulation and Schwinger models. J. Phys. A **31**, 2747–2760 (1998)
12. Banerjee, R., Rothe, H.J., Rothe, K.D.: Master equation for Lagrangian gauge symmetries. Phys. Lett. B **479**, 429–434 (2000)
13. Banerjee, R., Rothe, H.J., Rothe, K.D.: Recursive construction of generator for Lagrangian gauge symmetries. J. Phys. A **33**, 2059–2068 (2000). hep-th/9909039
14. Banerjee, R., Samanta, S.: Gauge generators, transformations and identities on a noncommutative space. Eur. Phys. J. C **51**, 207–215 (2007). hep-th/0608214
15. Weinberg, S.: Gravitation and Cosmology. Wiley, New York (1972)
16. Ortin, T.: Gravity and Strings. Cambridge University Press, Cambridge (2004)
17. Gegenberg, J., Kelly, P.F., Mann, R.B., Vincent, D.: Theories of gravitation in two dimensions. Phys. Rev. D **37**, 3463 (1988)
18. Mukhopadhyay, A., Padmanabhan, T.: Holography of gravitational action functionals. Phys. Rev. D **74**, 124023 (2006). hep-th/0608120
19. Exirifard, Q., Sheikh-Jabbari, M.M.: Lovelock gravity at the crossroads of Palatini and metric formulations. arXiv:0705.1879 [hep-th]
20. Carroll, S.M., Duvvuri, V., Trodden, M., Turner, M.S.: Is cosmic speed-up due to new gravitational physics? Phys. Rev. D **70**, 043528 (2004). astro-ph/0306438
21. Vollick, D.N.: 1/R Curvature corrections as the source of the cosmological acceleration. Phys. Rev. D **68**, 063510 (2003). astro-ph/0306630
22. Nojiri, S., Odintsov, S.D.: Modified gravity with negative and positive powers of the curvature: Unification of the inflation and of the cosmic acceleration. Phys. Rev. D **68**, 123512 (2003). hep-th/0307288
23. Vollick, D.N.: On the viability of the Palatini form of $1/R$ gravity. Class. Quantum Gravity **21**, 3813–3816 (2004). gr-qc/0312041
24. Flanagan, E.E.: Higher order gravity theories and scalar tensor theories. Class. Quantum Gravity **21**, 417–426 (2003). gr-qc/0309015
25. Landau, L.D., Lifshitz, E.M.: The Classical Theory of Fields, 4th edn. Pergamon, Elmsford (1994)